4D Curves and How to Visualize Them

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3D curves vs. 4D curves

Motivation: visualizing the higher dimensions

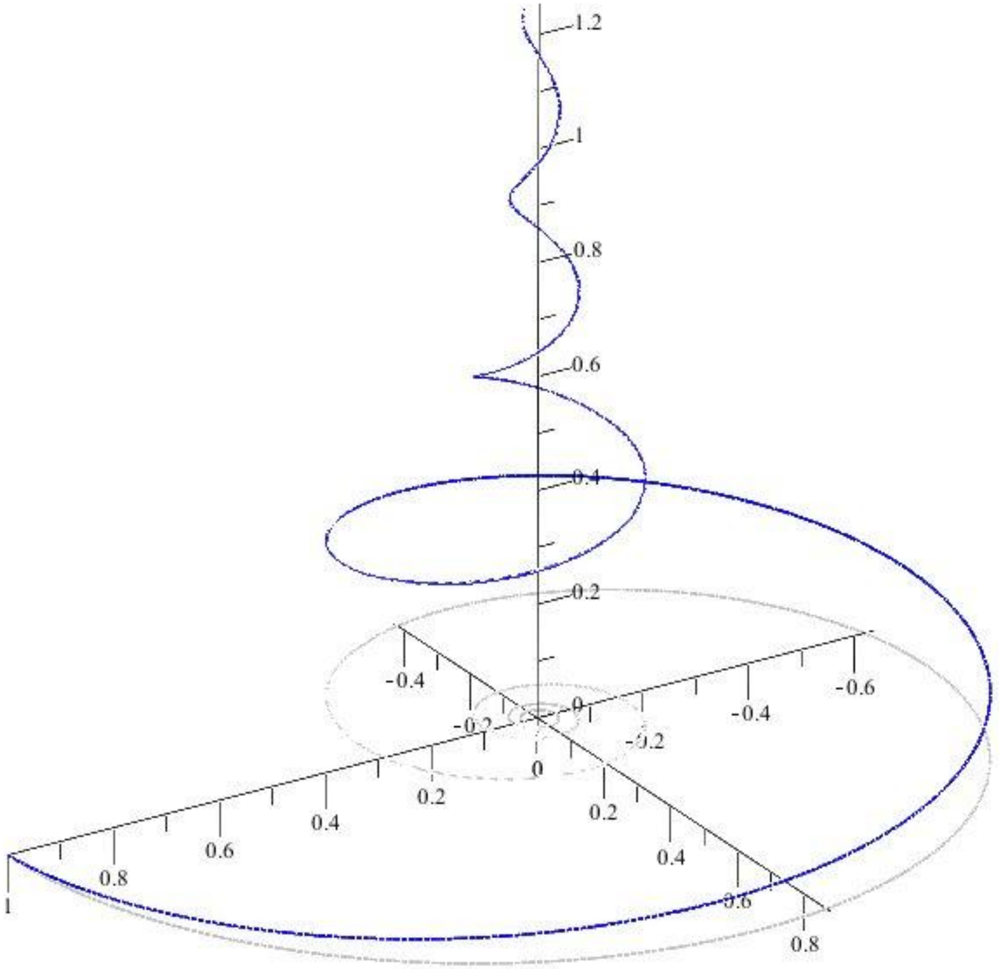
- Generalize 3D method to 4D
- Apply method to 4D helix
- Visualize in various ways

Curves Basics in 3D

3D Curves

$$\vec{\alpha} : \mathbb{R} \to \mathbb{R}^3$$

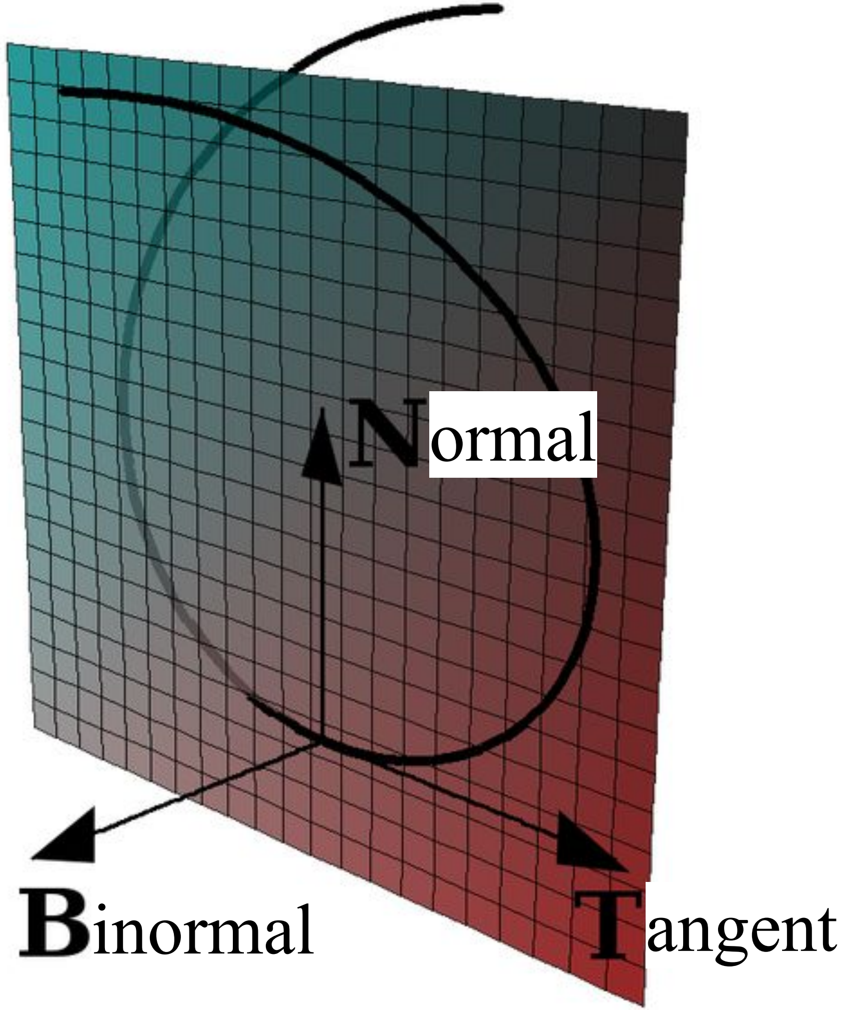
$$\vec{\alpha}(s) = (x(s), y(s), z(s))$$



“Frenet Formulas”

Curvatures

$$
\begin{aligned}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= -\tau N
\end{aligned}
$$



How can we take this to 4D?

4D Frenet Frame

$$
\begin{aligned}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= -\tau N + \xi Q \\
Q' &= -\xi B
\end{aligned}
$$

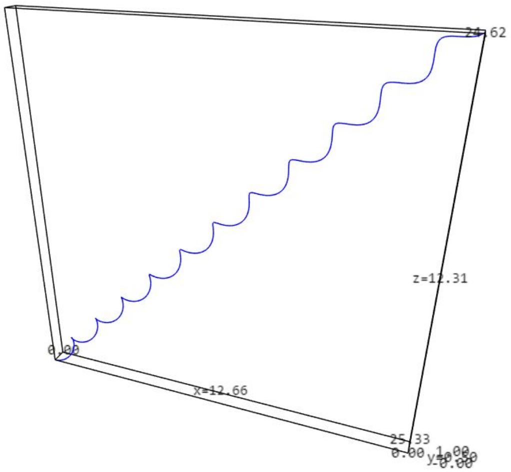
Quad = 4

Case Study: Helix

3D Helix

$$\kappa = 1$$

$$\tau = 1$$



4D Helix

$$\kappa = 1$$

$$\tau = 1$$

$$\xi = 1$$

?

First, get positions (α)

Solve Frenet formulas differential equations

Initial conditions:

$$\alpha(0) = (0, 0, 0, 0)$$
$$T(0) = (1, 0, 0, 0)$$
$$N(0) = (0, 1, 0, 0)$$
$$B(0) = (0, 0, 1, 0)$$
$$Q(0) = (0, 0, 0, 1)$$

Symbolic Computation with sage

```
-1/40*((sqrt(10)*(2*(22962338582515268997590264550714684762086198224853299108490864950321994009495351806694001252782243408752482
274760631686741985296961691714230746912763959754385701818438687372962219129176893255008410587822556635254346062511198098283857 6
963254077701633327840818601470041052414799606630476718336787670153037994368402833218174593584107730684343569153397966368197024 2
321*sqrt(5)*sqrt(2)*sqrt(-1/2*sqrt(5) + 3/2) - 51345349992870305348052442256932767909797402853196257965549375416050035490557637
555034328868788382924249963917644730210533003624096055835193076489219707787589474034140621073136051732521312328972027584295151 5
084325571203337320376502537451943385441328362335060676690320026523224244478387405523170145870387956309666817805849031262906032 4
479149063626695817954472761 98*sqrt(2)*sqrt(-1/2*sqrt(5) + 3/2))*sqrt(sqrt(5) + 3) + (17541665754675501644718351395211286376461 1
918213636393599232194349159465379284178650476748895583473020074843251742203917265552847546955917293308936990848439815805190200 0
011573052214278548532128029005038154754919782475500221091086547714456789456076714728894973674109115459207639544359141445804599 5
872075187474241817873938877626032588808239216823926940925046345076 5*sqrt(5)*sqrt(2)*sqrt(-sqrt(5) + 3) - 3922435706603457105620
600401722487991896121743532227835419380149654013642419615363163298034245393172898747801490260629449974602420358179404188382946 1
135643483331180487494571297198738719274236495553361757622762580368235441291198757876419312627124231869496425195656890745882969 2
401911326299730126929959264150508858792508817414583331563126566261418110426976490616989*sqrt(2)*sqrt(-sqrt(5) + 3))*sqrt(1/2*sq
rt(5) + 3/2)) - 4*(229623385825152689975902645507146847620861982248532991084908649503219940094953518066940012527822434087524827
476063168674198529696169171423074691276395975438570181843868737296221912917689325500841058782255663525434606251119809828385769 6
325407770163332784081860147004105241479960663047671833678767015303799436840283321817459358410773068434356915339796636819702423 2
1*sqrt(5)*sqrt(-1/2*sqrt(5) + 3/2) - 51345349992870305348052442256932767909797402853196257965549375416050035490557637555034328 8
687883829242499639176447302105330036240960558351930764892197077875894740341406210731360517325213123289720275842951515084325571 2
033373203765025374519433854413283623350606766903200265232224244478387405523170145870387956309666817805849031262906032447914906 36
266958179544727619 8*sqrt(-1/2*sqrt(5) + 3/2))*sqrt(sqrt(5) + 3) - 2*(1754166575467550164471835139521128637646119182136363935992
321943491594653792841786504767488955834730200748432517422039172655528475469559172933089369908484398158051902000115730522142785
485321280290050381547549197824755002210910865477144567894560767147288949736741091154592076395443591414458045995872075187474241 8
178739388776260325888082392168239269409250463450765*sqrt(5)*sqrt(-sqrt(5) + 3) - 3922435706603457105620600401722487991896121743
532227835419380149654013642419615363163298034245393172898747801490260629449974602420358179404188382946113564348333118048749457 1
297198738719274236495553361757622762580368235441291198757876419312627124231869496425195656890745882969240191132629973012692995 9
264150508858792508817414583331563126566261418110426976490616989*sqrt(-sqrt(5) + 3))*sqrt(1/2*sqrt(5) + 3/2))*sin(1/2*sqrt(2)*s*
sqrt(sqrt(5) + 3)) + (sqrt(10)*(175416657546755016447183513952112863764611918213636393599232194349159465379284178650476748895 58
```

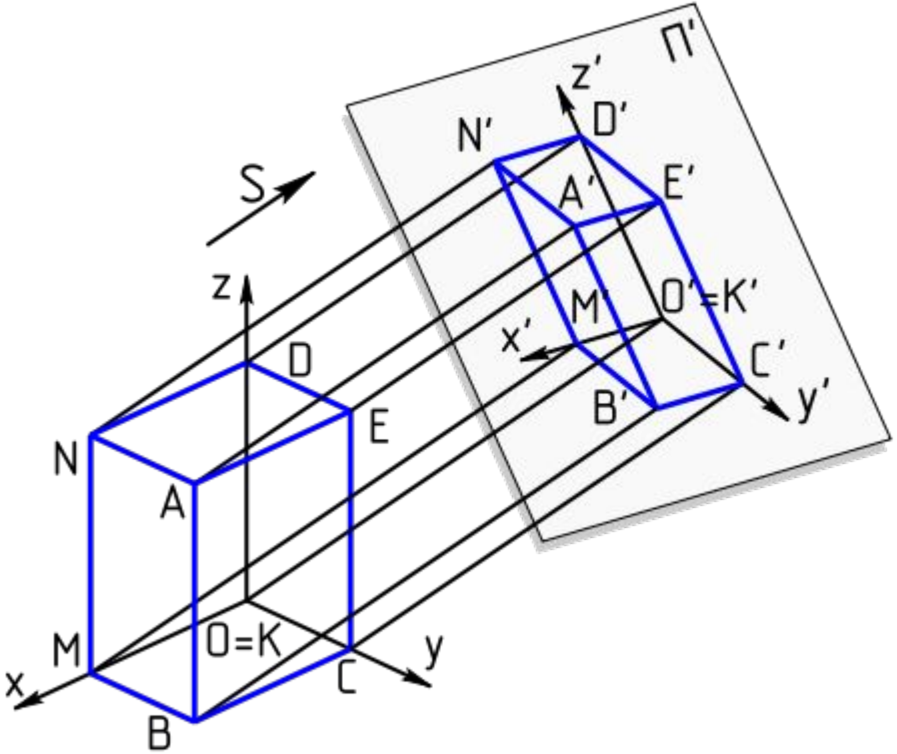
...

Numerical Integration (still with sage)

```
[[ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]
 [ 2.95540287e-01  4.43300481e-02  4.43959291e-03  3.34472398e-04]
 [ 5.65268584e-01  1.69518649e-01  3.40998621e-02  5.20889396e-03]
 ...
 [ 5.86628476e-01  1.36167910e+00  3.05968755e-01  2.19642803e+00]
 [ 4.63487787e-01  1.46542972e+00  6.01512345e-02  2.25097471e+00]
 [ 3.04547010e-01  1.59703629e+00 -1.55509262e-01  2.23556356e+00]]
```

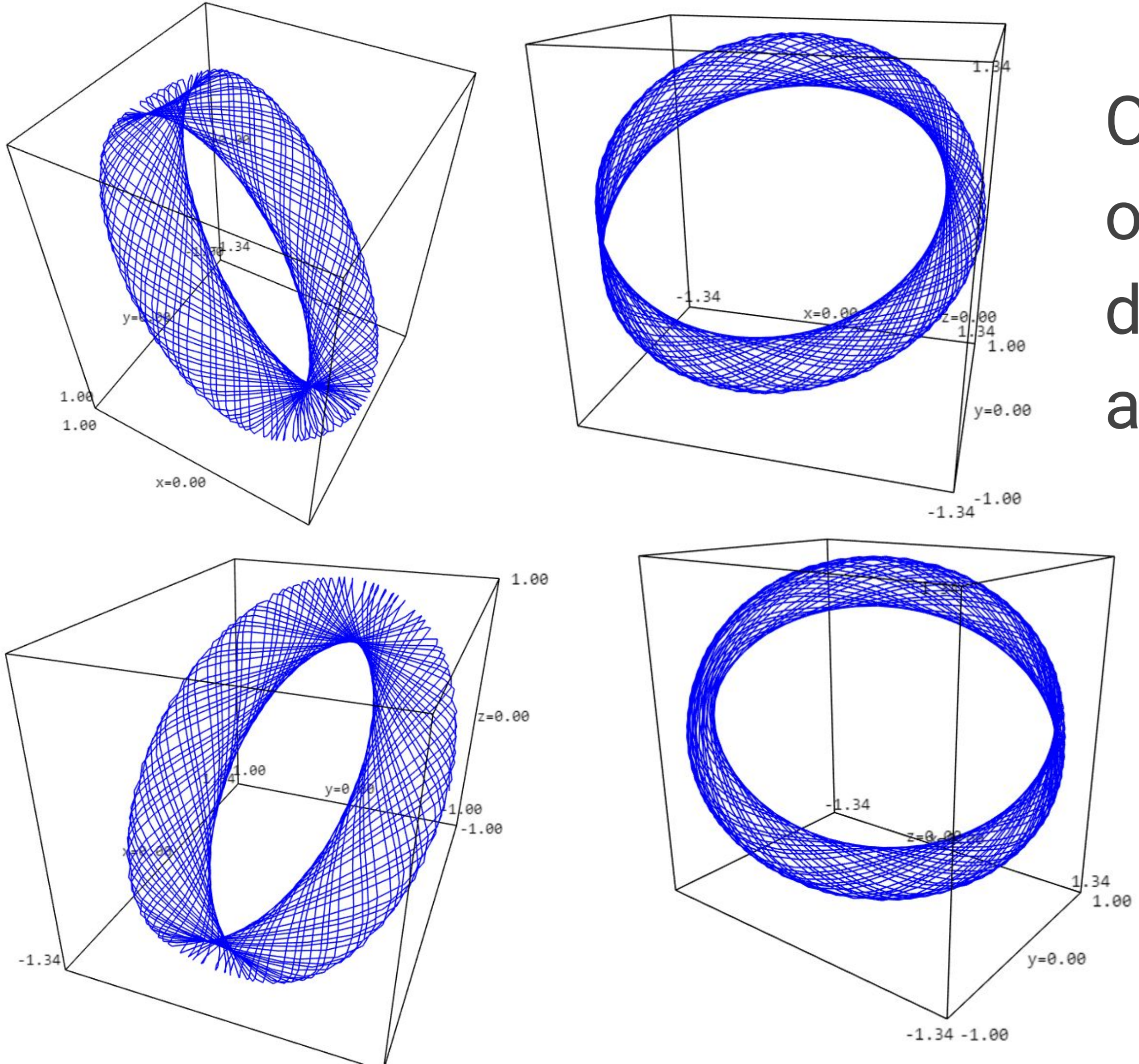
How to visualize this thing?

Technique 1: Orthogonal Projections



3D Object → 2D Shadows

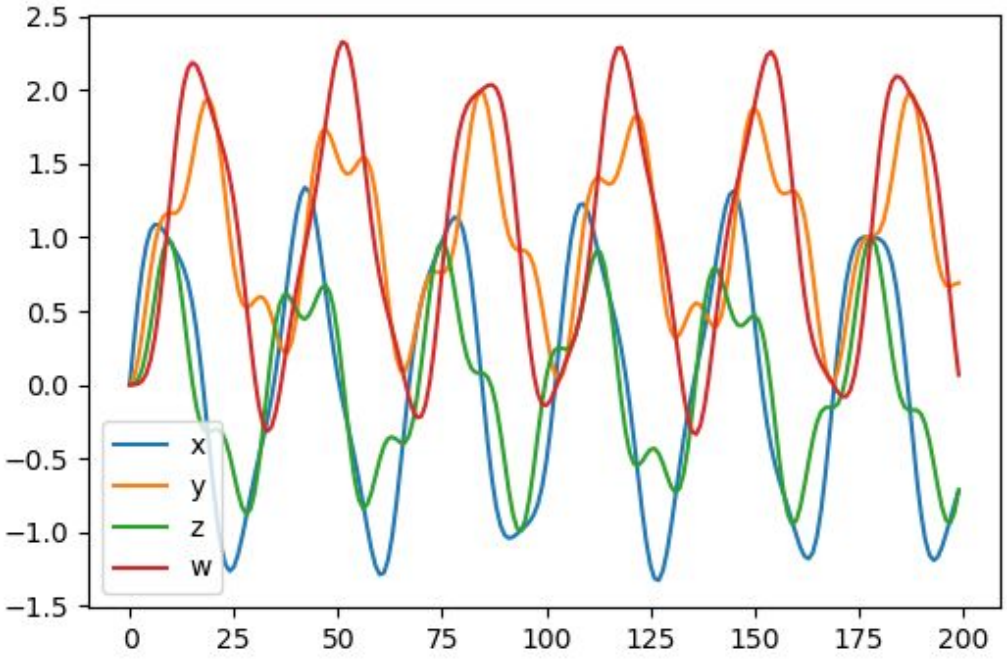
4D Object → 3D Shadows



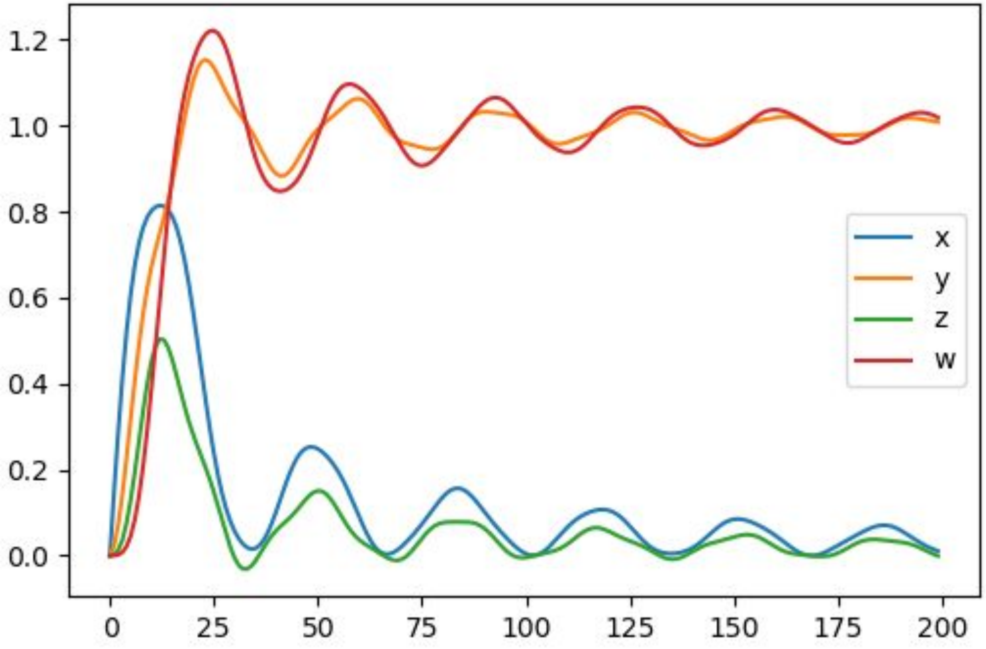
Only see 3 out of 4 dimensions at a time

Technique 2: Look at the Graph

4 dimensions
at once!



2.1: Center of Mass Plot

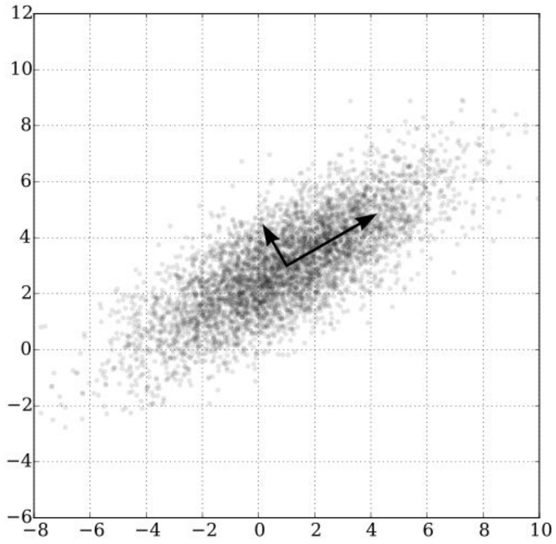


Bad initial conditions!

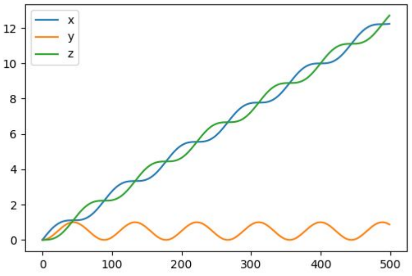
Technique 3: Principal Component Analysis (PCA)

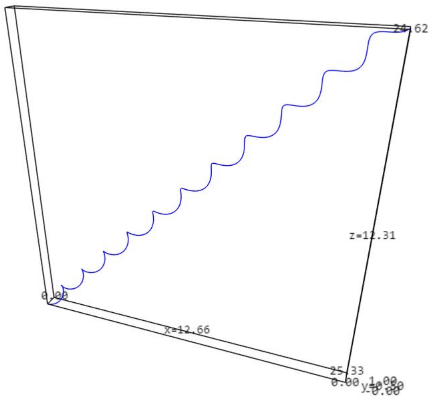
Finding a basis that **maximizes variances along the basis vectors**

→ A “nice” frame
to view the object from



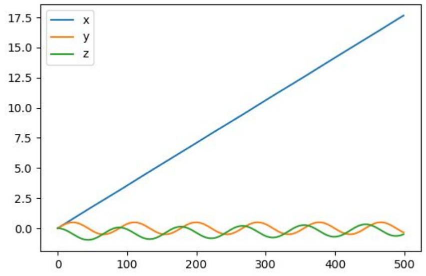
3D Example

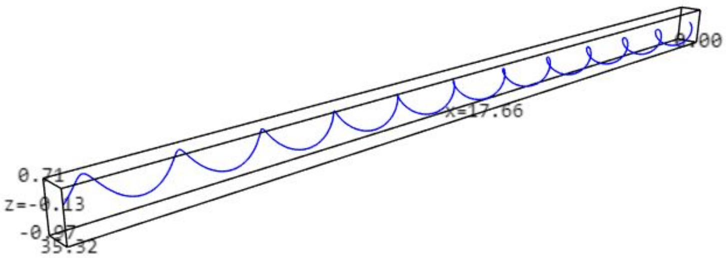




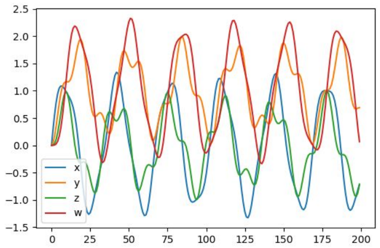
PCA

⇒



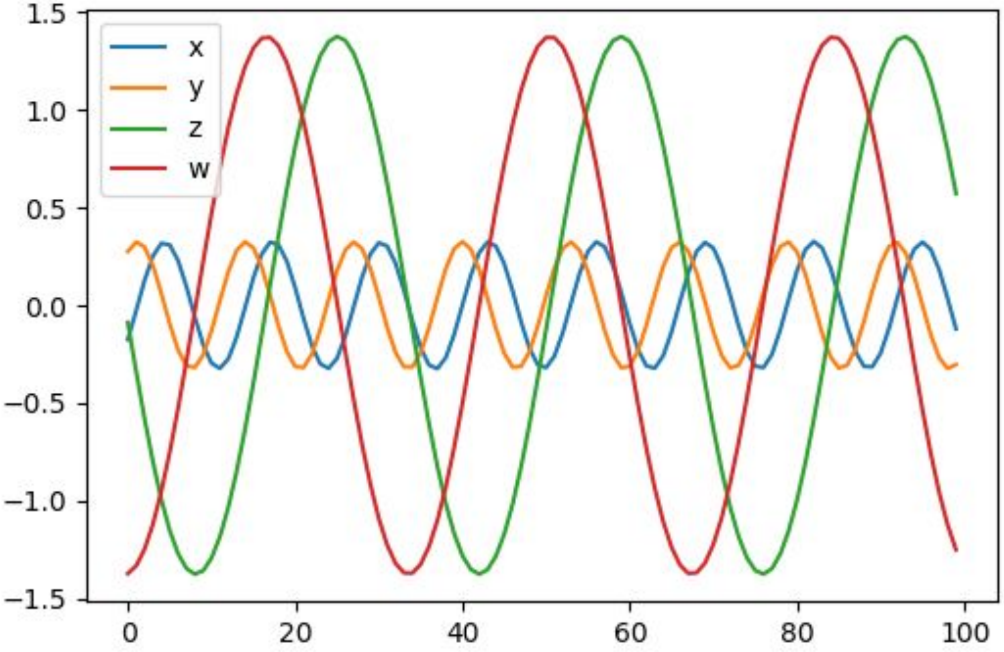


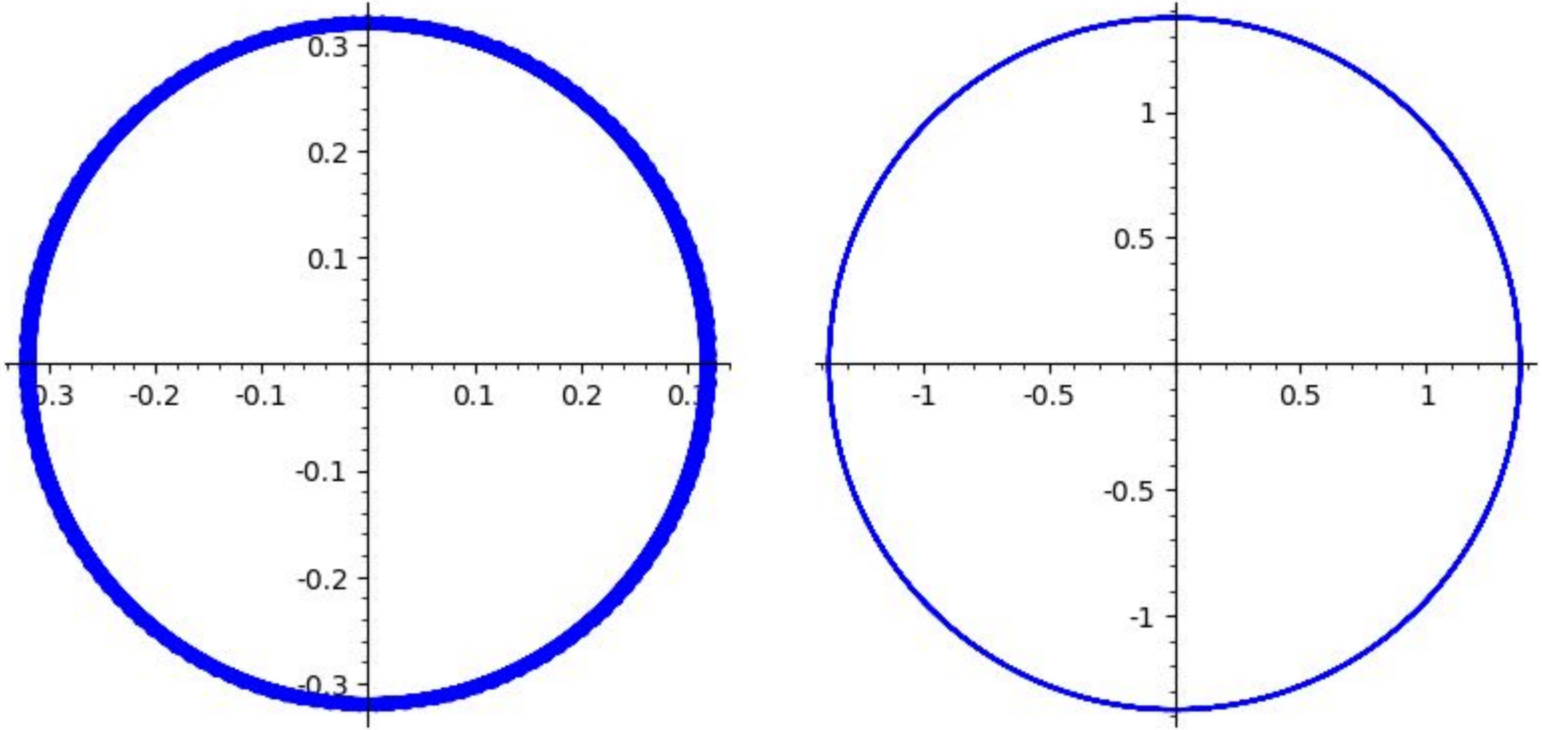
4D PCA



PCA
+
New IC's

⟹




0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.2
-0.1
0.1
0.2
1
0.5
-0.5
-1
-1
-0.5
0.5
1

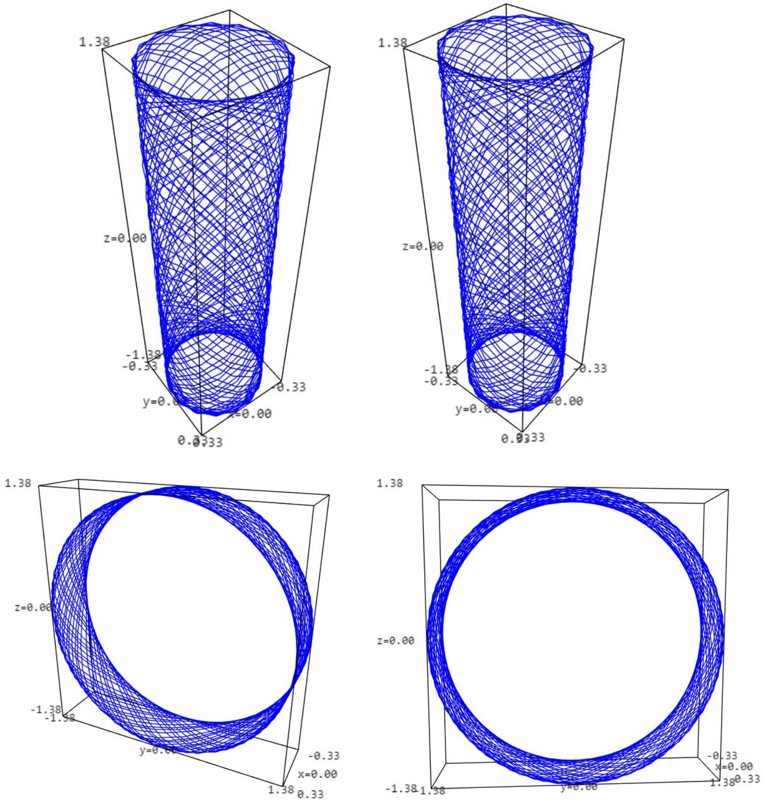
⇒ "Flat Torus"

$$\vec{x}(u, v) = (r_1 \cos(u), r_1 \sin(u), r_2 \cos(v), r_2 \sin(v))$$

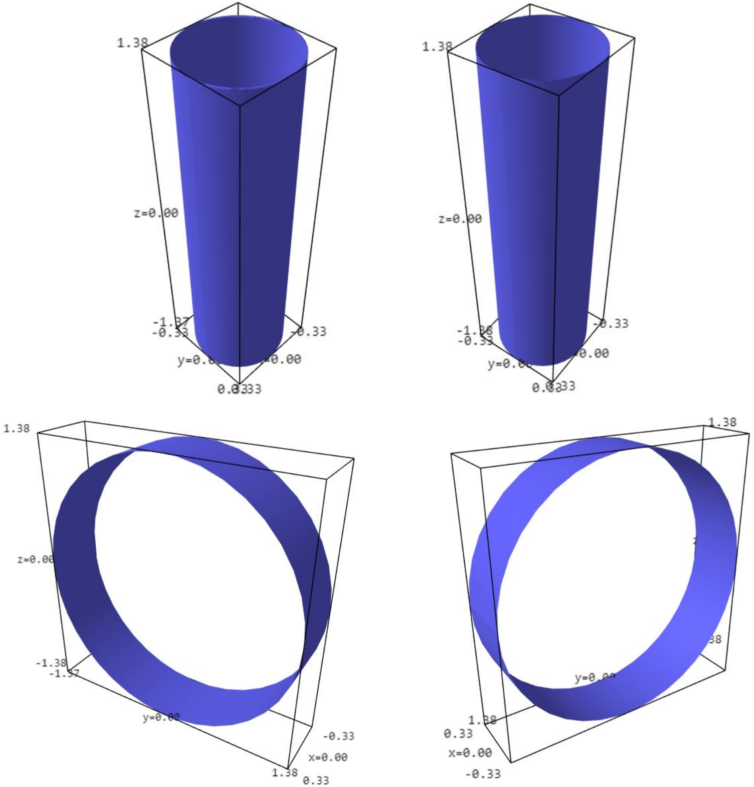
For us

- $r_1 = 0.327$
- $r_2 = 1.376$

α_{pca}



Flat Torus



Summary

Several numerical techniques to visualize 4D

- Numerical integration
- Orthogonal projections
- Graph plotting: positions and CMs
- PCA

Pros

- Computationally cheap
- Very visual
- Adaptation to a different curve is trivial

Cons

- Not mathematically rigorous
- Not guaranteed to work

Questions?